Assignment 1

Let Σ be a (finite) alphabet. Given a language $L \subseteq \Sigma^*$, define yolk(L) as follows:

$$\text{yolk}(L) := \{y \mid \exists x, z \in \Sigma^* \text{ such that } |x| = |y| = |z| \text{ and } xxyzz \in L\}.$$

Show that if L is recognizable, then yolk(L) is also recognizable.

Solution: Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be the DFA of L.

Consider the DFA $A_{(q_1,q_2,q_3,q_4)}$ =$(Q', \Sigma, \Delta', q_{1234}, F_{1234})$ Where Q' = Q^5 (5 cross product of the states of $\mathcal{A}$), $q_{1234} = (q_0, q_1, q_2, q_3, q_4)$, F_{1234} =$(q_1, q_2, q_3, q_4, q_f)$ where $q_f \in F$ The transition function δ' is defined as follows-

$((q_1, q_2, q_3, q_4, q_5), a, (q'_1, q'_2, q'_3, q'_4, q'_5)) \in \Delta'$

If and only if $\exists\ a', a'' \in \Sigma$ such that-

$\delta(q_3, a) = q'_3, \quad \delta(q_1, a') = q'_1, \quad \delta(q_2, a') = q'_2, \quad \delta(q_4, a'') = q'_4, \quad \delta(q_5, a'') = q'_5$,

> **Lemma:-** Suppose $y \in \Sigma^*$, and $((q_0, q_1, q_2, q_3, q_4), y, (q_5, q_6, q_7, q_8, q_9))$ $\in \Delta'$ iff there exist strings x and $z \in \Sigma^*$ of same length as y such such that:
>
> - $\delta(q_0, x) = q_5$, $\delta(q_1, x) = q_6$
> - $\delta(q_2, y) = q_7$
> - $\delta(q_3, z) = q_8$, $\delta(q_4, z) = q_9$
>
> **proof is by induction on the length of y:-** For base case $y = x = z = \varepsilon$(as there exists only 1 string of length 0 which is ε) the lemma holds, as $(((q_0, q_1, q_2, q_3, q_4), \varepsilon\ , (q_0, q_1, q_2, q_3, q_4)) \in \Delta'$ and
>
> - $\delta(q_0, \varepsilon) = q_0$, $\delta(q_1, \varepsilon) = q_1$
> - $\delta(q_2, \varepsilon) = q_2$
> - $\delta(q_3, \varepsilon) = q_3$, $\delta(q_4, \varepsilon) = q_4$
>
> Suppose the Lemma is true for length of $y = n$ $y = a_1 a_2 \dots a_n$, Then there exists $x = a'_1 a'_2 \dots a'_n$ and $z = a''_1 a''_2 \dots a''_n$,
>
> $((q_0, q_1, q_2, q_3, q_4), y, (q_5, q_6, q_7, q_8, q_9)) \in \Delta'$ where-
>
> $\delta(q_0, x) = q_5$, $\delta(q_1, x) = q_6$, $\delta(q_2, y) = q_7$, $\delta(q_3, z) = q_8$, $\delta(q_4, z) = q_9$.

Now suppose $(((q_5, q_6, q_7, q_8, q_9), a_{n+1}, (q'_5, q'_6, q'_7, q'_8, q'_9)) \in \Delta'$ (Note: Since it's an NFA, there may be multiple such states, but the inductive step is unaffected.)

Then there exist a', a'' such that:

$$\delta(q_5, a') = q'_5, \quad \delta(q_6, a') = q'_6, \quad \delta(q_7, w_{n+1}) = q'_7, \quad \delta(q_8, a'') = q'_8, \quad \delta(q_9, a'') = q'_9.$$

Hence, there exist strings $a'_1 \dots a'_n a'$ and $a''_1 \dots a''_n a''$ such that:

- $\delta(q_0, xa') = q'_5$, $\delta(q_1, xa') = q'_6$
- $\delta(q_2, ya_{n+1}) = q'_7$
- $\delta(q_3, za'') = q'_8$, $\delta(q_4, za'') = q'_9$

Define: $L_{(q_1,q_2,q_3,q_4)} = L(A_{(q_1,q_2,q_3,q_4)})$ and $L' = \bigcup_{(q_1,q_2,q_3,q_4) \in Q^4} L_{(q_1,q_2,q_3,q_4)}$

Main Claim:

$$L' = \text{yolk}(L)$$

Proof:

(i) $L' \subseteq \text{yolk}(L)$

We will show that $L_{(q_1,q_2,q_3,q_4)} \subseteq \text{yolk}(L)$ for any tuple (q_1, q_2, q_3, q_4). Then the whole union itself is also a subset of yolk(L).

Suppose $y \in L_{(q_1,q_2,q_3,q_4)}$ and the length of y is n. We will show that there exist x and z of the same length as y such that $xxyzz \in L$.

Therefore, if $((q_0, q_1, q_2, q_3, q_4), y, (q_1, q_2, q_3, q_4, q_f)) \in \Delta'$,then there exist x and z of the same length as y such that - $\delta(q_0, x) = q_1$, $\delta(q_1, x) = q_2$, $\delta(q_2, y) = q_3$, $\delta(q_3, z) = q_4$, $\delta(q_4, z) = q_f$, hence $\delta(q_0, xxyzz) = q_f$, which gives-

$$xxyzz \in L$$

So $y \in \text{yolk}(L)$.

(ii) yolk$(L) \subseteq L'$

So, suppose $y \in \text{yolk}(L)$. Then there exist x and z of the same length as y such that $xxyzz \in L$ and $\delta(q_0, x) = q_1$, $\delta(q_1, x) = q_2$, $\delta(q_2, y) = q_3$, $\delta(q_3, z) = q_4$, $\delta(q_4, z) = q_f$

Let n be the length of x, y, and z.

Suppose:

$$x = a'_1 a'_2 \dots a'_n, \quad y = a_1 a_2 \dots a_n, \quad z = a''_1 a''_2 \dots a''_n$$

Define the following state transitions:

$q_{1,i} = \delta\ (q_0, a'_1 a'_2 \dots a'_i)$
$q_{2,i} = \delta\ (q_1, a'_1 a'_2 \dots a'_i)$
$q_{3,i} = \delta\ (q_2, a_1 a_2 \dots a_i)$
$q_{4,i} = \delta\ (q_3, a''_1 a''_2 \dots a''_i)$
$q_{5,i} = \delta\ (q_4, a''_1 a''_2 \dots a''_i)$
for $i \in (1, 2, ...n)$

At $i = n$, we have:

$$q_{1,n} = q_1, \quad q_{2,n} = q_2, \quad q_{3,n} = q_3, \quad q_{4,n} = q_4, \quad q_{5,n} = q_f$$

Suppose ($(q_0, q_1, q_2, q_3, q_4)$, $a_1 a_2 \dots a_i$, $(q_{1,i}, q_{2,i}, q_{3,i}, q_{4,i}, q_{5,i})) \in \Delta'$
now $((q_{1,i}, q_{2,i}, q_{3,i}, q_{4,i}, q_{5,i}), y_{i+1}, (q_{1,i+1}, q_{2,i+1}, q_{3,i+1}, q_{4,i+1}, q_{5,i+1})) \epsilon \Delta'$,because there exist a'_{i+1} and a''_{i+1} such that:

$$\begin{aligned}
\delta(q_{1,i}, a'_{i+1}) &= q_{1,i+1} \\
\delta(q_{2,i}, a'_{i+1}) &= q_{2,i+1} \\
\delta(q_{3,i}, a_{i+1}) &= q_{3,i+1} \\
\delta(q_{4,i}, a''_{i+1}) &= q_{4,i+1} \\
\delta(q_{5,i}, a''_{i+1}) &= q_{5,i+1}
\end{aligned}$$

Hence, inductively after n steps-

$$((q_0, q_1, q_2, q_3, q_4), a_1 a_2 \dots a_n, (q_1, q_2, q_3, q_4, q_f)) \in \Delta'$$

Therefore, $y \in L_{(q_1, q_2, q_3, q_4)}$. **....Hence proven.**