TOC : Tutorial 2

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6th October 2025

Question 1: Let L be a regular language. Given a deterministic finite automaton $(Q, \Sigma, \delta, q_0, F)$ for L, construct an automaton accepting $L' = \{w | w^n \in L\}$, where n is a fixed positive integer.

We defined an equivalence relation on the set of words, $w_1 \equiv w_2$ iff for all $q \in Q, \hat{\delta}(q, w_1) = \hat{\delta}(q, w_2)$, where $\hat{\delta}$ is the transition function extended to words. From the definition, we can identify these with maps from Q to Q (the map is well defined since the automaton is deterministic). Since there are finitely many such maps (Q is a finite set), there are finitely many equivalence classes. Let A be the set of equivalence classes. Note that $|A| = |Q|^{|Q|}$. For a word w, let ψ_w be its equivalence class. Note that $\psi_{wa} = \psi_a \circ \psi_w$ for any $a \in \Sigma$. Also, note that ψ_ϵ is the identity function.

We define the automaton $(Q', \Sigma', \delta', q_0', F')$ where

$$Q' = A$$

$$\Sigma' = \Sigma$$

$$\delta'(\phi, a) = \psi_a \circ \phi$$

$$q_0' = \psi_\epsilon$$

$$F' = \{\psi \in A | \psi^n(q_0) \in F\}$$

Constructing this automaton takes $O(|Q|^{|Q|})$ time, which is exponential in the input. However, we saw that if we have to simply test membership in L', we can do it in polynomial time by just 'tracing' the path of w in our new automaton instead of constructing it fully.

Question 2: Let L be a regular language. Given a deterministic finite automaton $(Q, \Sigma, \delta, q_0, F)$ for L, construct an automaton accepting $L' = \{w | w^{|w|} \in L\}$

(Solution by Sayeed) We need to somehow store the 'length' of the word in finite memory. Let B be the set of maps from A to A. Note that $|B| = |A|^{|A|} = |Q|^{|Q|^3}$ Now, we construct the automaton $(Q', \Sigma', \delta', q_0', F')$ where

$$Q' = A \times B$$

$$\Sigma' = \Sigma$$

If we reach a state (ϕ, μ) on reading a word w, we want ϕ to be the equivalence class of w and μ to be the map $\phi' \mapsto \phi'^{|w|}$ (*).

So, we define the transition function as follows:

$$\delta'((\phi, \mu), a) = (\psi_a \circ \phi, \mu')$$

where μ' is the map $(\phi \mapsto (\mu(\phi)) \circ \phi)$. We set the other parameters as follows:

$$q_0' = (\psi_\epsilon, (\phi \mapsto \psi_\epsilon))$$

$$F' = \{(\phi, \mu) | (\mu(\phi))(q_0) \in F\}$$

Show that (*) is satisfied.

Note that this takes $\mathcal{O}(|Q|^{|Q|^{|Q|^{|Q|}}})$ time to compute.

An alternate solution:

For an element $\phi \in A$, consider the set $\{\phi^n | n \geq 0\}$. This is a subset of A, so it is finite. Calculate smallest n such that $\phi^n = \phi^m, m \neq n$. We call this n_ϕ. Let $N = lcm_{\phi \in A}\{n_\phi\}$. Now, we construct the automaton $(Q', \Sigma', \delta', q_0', F')$ where

$$Q' = A \times \{0, 1 \ldots (N-1)\}$$

$$\Sigma' = \Sigma$$

$$\delta'((\phi, n), a) = (\psi_a \circ \phi, (n+1) mod N)$$

$$q_0' = (\psi_\epsilon, 0)$$

$$F' = \{(\phi, n) | \phi^n\}$$

Note that this takes $\mathcal{O}(|Q|^{|Q|^{|Q|}})$ time to compute, since N is bounded by $|Q|^{|Q|^{|Q|}}$.

Question 3: Let L be a regular language. Given a deterministic finite automaton $(Q, \Sigma, \delta, q_0, F)$ for L, construct an automaton accepting $L' = \{w | w^{p(|w|)} \in L\}, where pis somepolynomial.$

The first solution of the previous question can be generalised to this question. Suppose $p(x) = \Sigma_{i=0}^{n} a_i x^i$. Then, we modify F' to be

$$F' = \{(\phi, \mu) | ((\mu^n(\phi^{a_n})) \circ (\mu^{n-1}(\phi^{a_{n-1}})) \circ \ldots \mu(\phi^{a_1}) \circ \phi^{a_0})(q_0) \in F\}$$

This is still $\mathcal{O}(|Q|^{|Q|^{|Q|^{|Q|}}})$ time.

Can you give a $\mathcal{O}(|Q|^{|Q|^{|Q|}})$ time solution by generalising the second solution of the previous problem?