1 Dealing with binary numbers in Finite Automata

1. Let $\Sigma = \{0, 1\}$, and consider the language consisting of all words over Σ which represent, in binary, numbers that are congruent to 1 (mod 3). That is, the language is $\{\, 1,\ 100,\ 111,\ \ldots\}$. Construct an NFA accepting this language. (Note: leading zeros are allowed. i.e 0^*1 should be accepted.)

2. Construct an NFA that verifies binary addition of numbers in the following way. Let $\Sigma = \{0, 1\}^3$. Given a word w over Σ, let w_i be the word over $\{0, 1\}$ constructed by taking the ith component of each letter in w. Let L be a language over Σ such that

$$L = \{\, w \mid w_1 + w_2 = w_3, \text{ where the words are read as binary numbers} \,\}.$$

Example. Consider $w = (0, 0, 1)(1, 1, 0)(0, 1, 1)(0, 1, 1)$. Then $w_1 = 0100$, $w_2 = 0111$, and $w_3 = 1011$, and indeed $w_1 + w_2 = w_3$. Hence $w \in L$. Give an NFA for L.

3. **(1.38)** Let

$$\Sigma_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Here, Σ_2 contains all columns of 0s and 1s of height two. A string of symbols in Σ_2 gives two rows of 0s and 1s. Consider each row to be a binary number and let

$$C = \{\, w \in \Sigma_2^* \mid \text{the bottom row of } w \text{ is three times the top row} \,\}.$$

For example, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in C$, but $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin C$. Show that C is regular.

4. **(1.39)** Let Σ_2 be the same as in Problem 1.38. Consider each row to be a binary number and let

$$D = \{\, w \in \Sigma_2^* \mid \text{the top row of } w \text{ is a larger number than the bottom row} \,\}.$$

2 Question on Closure Properties

5. Let L be a regular language, and let

$$L' = \{\, w \mid ww \in L \,\}.$$

Prove that L' is regular.

6. If A is any language, let $A_{\frac{1}{3}-\frac{1}{3}}$ be the set of all strings in A with their middle thirds removed so that

$$A_{\frac{1}{3}-\frac{1}{3}} = \{\, xz \mid \text{for some } y,\ |x| = |y| = |z| \text{ and } xyz \in A \,\}.$$

Show that if A is regular, then $A_{\frac{1}{3}-\frac{1}{3}}$ is not necessarily regular.

7. For a language L, we define

$$\begin{aligned}\text{base}(L) = \{\, w \in \Sigma^+ \ \mid\ & w = a_1 a_2 \dots a_m \text{ for some } m \in \{1, 2, \dots\}, \\ & a_i \in \Sigma,\ \exists n_1, \dots, n_m \in \{1, 2, \dots\}, \\ & a_1^{n_1} a_2^{n_2} \dots a_m^{n_m} \in L \,\}.\end{aligned}$$

That is, base(L) contains words which are obtained from words in L by deleting an arbitrary number of letters, but not everything, from a contiguous block consisting of the same letter. If L is regular then base(L) is also regular. Describe how to construct an NFA for base(L), given an NFA for L.

8. Let $L_1, L_2 \subseteq \Sigma^*$ be two regular languages. Recall that for a language $L \subseteq \Sigma^*$ and a string $w \in \Sigma^*$, the *left quotient* of L by w is defined as

$$w^{-1}L = \{\, x \in \Sigma^* \mid wx \in L \,\}.$$

For a language L_1, we extend this to

$$L_1^{-1}L_2 = \bigcup_{w \in L_1} w^{-1}L_2.$$

(a) Prove or disprove: if L_1 and L_2 are regular languages, then $L_1^{-1}L_2$ is also regular.

(b) If the statement is true, describe how to construct an automaton for $L_1^{-1}L_2$, given NFAs for L_1 and L_2.

(c) is $L_1^{-1}L_2$ still regular is L_1 is an arbitrary language.