Chennai Mathematical Institute

Design and Analysis of Algorithms CU2101

Assignment-3

Aug-Nov 2025

1. Following set of questions asks you to develop efficient algorithms to find optimal subsequences of various kinds. (20)

   (a) Let $A[1..m]$ and $B[1..n]$ be two arbitrary arrays. A *common supersequence* of A and B is another sequence that contains both A and B as subsequences. Describe an efficient algorithm to compute the length of the **shortest common supersequence** of A and B.

   (b) Call a sequence $X[1..n]$ of numbers *bitonic* if there is an index i with $1 < i < n$, such that the prefix $X[1..i]$ is increasing and the suffix $X[i..n]$ is decreasing. Describe an efficient algorithm to compute the length of the **longest bitonic subsequence** of an arbitrary array A of integers.

   (c) Call a sequence $X[1..n]$ of numbers *oscillating* if $X[i] < X[i+1]$ for all even i, and $X[i] > X[i+1]$ for all odd i. Describe an efficient algorithm to compute the length of the **longest oscillating subsequence** of an arbitrary array A of integers.

   (d) Call a sequence $X[1..n]$ of numbers *convex* if $2X[i] < X[i-1] + X[i+1]$ for all i. Describe an efficient algorithm to compute the length of the **longest convex subsequence** of an arbitrary array A of integers.

2. Let P be a set of n points evenly distributed on the unit circle, and let S be a set of m line segments with endpoints in P. The endpoints of the m segments are not necessarily distinct; n could be significantly smaller than $2m$. (20)

   (a) Describe an algorithm to find the size of the largest subset of segments in S such that every pair is **disjoint**. Two segments are disjoint if they do not intersect even at their endpoints.

   (b) Describe an algorithm to find the size of the largest subset of segments in S such that every pair is **interior-disjoint**. Two segments are interior-disjoint if their intersection is either empty or an endpoint of both segments.

   (c) Describe an algorithm to find the size of the largest subset of segments in S such that every pair **intersects**.

   (d) Describe an algorithm to find the size of the largest subset of segments in S such that every pair **crosses**. Two segments cross if they intersect but not at their endpoints.

For full credit, all four algorithms should run in $O(mn)$ time.

3. Consider the following process. At all times you have a single positive integer x, which is initially equal to 1. In each step, you can either **increment** x or **double** x. Your goal is to produce a target value n.

   For example, you can produce the integer 10 in four steps as follows:

   $$1 \xrightarrow{+1} 2 \xrightarrow{\times 2} 4 \xrightarrow{+1} 5 \xrightarrow{\times 2} 10.$$

   Obviously, you can produce any integer n using exactly $n-1$ increments, but for almost all values of n, this is horribly inefficient.

   Describe and analyze an algorithm to compute the **minimum number of steps** required to produce any given integer n. (10)

4. In each of the following cases, prove that the given set system (S, I) satisfies the axioms of a **matroid**. That is, show that I is nonempty, hereditary (closed under subsets), and satisfies the augmentation (exchange) property. (15)

   (a) Let T be an $m \times n$ matrix over some field (for example, the real numbers). Let S be the set of columns of T, and define

   $$I = \{A \subseteq S \mid \text{the columns in } A \text{ are linearly independent}\}.$$

   Show that (S, I) is a matroid.

   (b) Suppose (S, I) is a matroid. Define

   $$I' = \{A' \subseteq S \mid S \setminus A' \text{ contains some maximal } A \in I\}.$$

   That is, the maximal independent sets of (S, I') are precisely the complements of the maximal independent sets of (S, I). Show that (S, I') is a matroid.

   (c) Let S be a finite set, and let $S_1, S_2, \ldots, S_k$ be a partition of S into nonempty, disjoint subsets. Define

   $$I = \{A \subseteq S \mid |A \cap S_i| \leq 1 \text{ for all } i = 1, 2, \ldots, k\}.$$

   Show that (S, I) is a matroid. (That is, I consists of all subsets that contain at most one element from each S_i.)

5. Let $G = (V, E)$ be a weighted, directed graph with a nonnegative weight function $w : E \to \{0, 1, \ldots, W\}$ for some nonnegative integer W.

   Modify Dijkstra's algorithm to compute the **shortest paths** from a given source vertex s in $O(W|V| + |E|)$ time. (10)

6. Prove that a binary tree that is not full cannot correspond to an optimal prefix code. (5)