Calculus II, CMI. *

Poincaré lemma. [1]

Consider an open domain $U \subset \mathbb{R}^n$ with coordinates $x^1, \dots, x^n$. Denote by $\Lambda T^*(U) = \oplus_m \Lambda^m T^*(U)$ the graded commutative algebra of all differential forms on U. Recall that $\Lambda^0 T^*(U) = C^\infty(U)$ is the space of all smooth functions and $d\colon \Lambda T^*(U) \to \Lambda T^*(U)$ is the exterior differential such that $d \circ d = 0$. We say that d has degree 1 because its components rise the degree $d\colon \Lambda^m T^*(U) \to \Lambda^{m+1} T^*(U)$.

For a vector field $v \in T(U)$, define the *contraction map* $\iota_v\colon \Lambda^m T^*(U) \to \Lambda^{m-1} T^*(U)$ by setting

$$(\iota_v \omega)(v_1, \dots, v_{m-1}) = \omega(v, v_1, \dots, v_{m-1}),$$

for all vector fields $v_1, \dots, v_{m-1} \in T(U)$. The contraction ι_v has degree -1. In the previous exercises we saw that ι_v is a derivation.[2]

Working in the graded setting we define *the (super) commutator* of two abstract endomorphisms $A_i, i = 1, 2$ of degrees m_i by

$$[A_1, A_2] = A_1 \circ A_2 - (-1)^{m_1 m_2} A_2 \circ A_1,$$

where m_i is the degree of A_i.

Problem 1 Assume $A_i, i = 1, 2, 3$ are endomorphisms of degrees m_i, respectively. Prove that the bracket $[-, -]$ satisfy the Jacobi identity

$$[A_1, [A_2, A_3]] = [[A_1, A_2], A_3] + (-1)^{m_1 m_2} [A_2, [A_1, A_3]].$$

In other words the operator $[A_1, -]$ is a derivation with respect to the bracket $[-, -]$.

Definition. The *Lie derivative* $\mathcal{L}_v\colon \Lambda T^*(U) \to \Lambda T^*(U)$ along the vector field $v \in T(U)$, is the commutator

$$\mathcal{L}_v := [\iota_v, d] = \iota_v \circ d + d \circ \iota_v.$$

Problem 2 Check that $\mathcal{L}_v$ is a derivative of degree 0, i.e.:

$$\mathcal{L}_v(\omega_1 \wedge \omega_2) = \mathcal{L}_v(\omega_1) \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_v(\omega_2),$$

for all differential forms ω_1, ω_2.

Hint: Prove that the (super) commutator of two derivatives is a derivative.

Problem 3 Show that $\mathcal{L}_v$ commutes with d, i.e. $[\mathcal{L}_v, d] = \mathcal{L}_v \circ d - d \circ \mathcal{L}_v = 0$.

Problem 4 Assume $L_v\colon \Lambda T^*(U) \to \Lambda T^*(U)$ is a derivation of degree 0 such that $[L_v, d] = 0$ and L_v restricted to 0-forms $C^\infty(U) \to C^\infty(U)$ is the usual derivative of a function along v. Prove that $L_v = \mathcal{L}_v$.

Problem 5 Assume $v, w \in T(U)$ are two vector fields. Show that there is unique vector field $u \in T(U)$ such that that $[\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_u$. Define the *commutator* of v and w to be $[v, w] := u$. Check that $[v, fw] = \mathcal{L}_v(f)w + f[v, w]$ and deduce the formula

$$[v^i \partial_i, w^j \partial_j] = v^i \partial_i(w^j) - w^j \partial_j(v^i),$$

for all functions v^i, w^j.

Problem 6' Prove that $[\mathcal{L}_v, \iota_w] = \iota_{[v,w]}$.

Hint: Both parts are derivations of forms of degree -1. It remains to compare the actions on each side on 1-forms df for various functions f.

Problem 7' Assume $\nu \in \Lambda^1 T^*(U)$, prove the following intrisic expression for the de Rham differential:

$$(d\nu)(v_1, v_2) = \mathcal{L}_{v_1}(\nu(v_2)) - \mathcal{L}_{v_2}(\nu(v_1)) - \nu([v_1, v_2]).$$

[3]

Hint: Denote the difference of both parts by $B(\nu, v_1, v_2)$. Check that B is $C^\infty(U)$-linear in all arguments, e.g. $B(f\nu, v_1, v_2) = fB(\nu, v_1, v_2)$ for all $f \in C^\infty(U)$. Reduce to the case of constant vector fields and forms, where B clearly vanish.

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*Mistakes are to be sent to Alexander Zakharov, zakharov@cmi.ac.in; Date 31/10/25.

[1] It is safe to read the intro and then move immediately to Problems 8 and 9.

[2] By the definition it means that the Leibniz identity holds: $\iota_v(\omega_1 \wedge \omega_2) = \iota_v(\omega_1) \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \iota_v(\omega_2)$. The general convention is to introduce the sign equal to $(-1)^{m_1 m_2}$ each time two objects of degrees m_1, m_2 are passing through each other.

[3] One interpretation of this formula applied to constant vector fields tells that $(d\nu)(v_1, v_2)$ evaluated at point p is equal to $\lim\limits_{\varepsilon \to 0} \frac{\int_{\partial \Pi_{\varepsilon\, v_1, \varepsilon\, v_2}} \nu}{\varepsilon^2}$, where $\partial \Pi_{\varepsilon\, v_1, \varepsilon\, v_2}$ is the boundary of the parallelogram with sides $\varepsilon\, v_1, \varepsilon\, v_2$ at the given point p. This is the infinitisimal form Stokes' theorem.

Let $I = [0,1]$ be the unit interval with the coordinate t. Consider the projection $p\colon U \times I \to U$ with fibers equal to I. Denote by $\partial_t \in T(U \times I)$ the vector field along the fibers of p. Define *the integration along the fibers of* p to be the map $p_*\colon \Lambda^m T^*(U \times I) \to \Lambda^{m-1} T_*(U)$ given by the formula

$$(p_*\omega)(v_1, \dots, v_{m-1}) = \int_0^1 \omega(\partial_t, v_1, \dots, v_{m-1})dt,$$

for all vector fields $v_1, \dots, v_{m-1} \in T(U)$. Concretely, if $\omega = \alpha_\bullet dx^\bullet + \beta_\bullet dt dx^\bullet$, then $p_*\omega = \left(\int_0^1 \beta_\bullet dt\right) \cdot dx^\bullet$. [4] Denote by $i_t\colon U \to U \times I$ the inclusion of $U = U \times \{t\}$.

Problem 8 Check that for all $\omega \in \Lambda^m T^*(U \times I)$ we have

$$d(p_*(\omega)) + p_*(d(\omega)) = i_1^*\omega - i_0^*\omega.$$

Let $U' \subset \mathbb{R}^m$ be an open domain provided with a smooth map $G\colon U \times I \to U'$. Let $H := p_* \circ G^*$. It is a map from $\Lambda^m T^*(U')$ to $\Lambda^{m-1} T^*(U)$ for all m.

Problem 9 *(The homotopy lemma)* Let $G_t^* = i_t^* \circ G^*$. Prove the equality

$$G_1^* - G_0^* = d \circ H + H \circ d$$

of maps from $\Lambda T^*(U')$ to $\Lambda T^*(U)$. [5]

The de Rham differential components $d^i\colon \Lambda^i T^*(U) \to \Lambda^{i+1} T^*(U)$ satisfy $d^i \circ d^{i-1} = 0$, which is equivalent to the inclusion $\operatorname{Im} d^{i-1} \subset \operatorname{Ker} d^i$ of exact forms to the closed ones.

Definition. The i-th *de Rham cohomology* of U is the real vector space $H^i(U) = \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$ equal to the quotient of closed i-forms modulo exact i-forms.

Problem 10 (*Poincaré lemma)* Assume U is *contractible*, i.e. there is a smooth map $G\colon U \times I \to U$ such that $G_0 = \mathrm{id}_U$ and G_1 is a constant map. Show that $H^0(U) = \mathbb{R}$ and $H^i(U) = 0$ for all $i > 0$. In particular, the cohomology of $\mathbb{R}^n$ vanish except of degree 0.

Let $\omega \in \Lambda^n T^*(U)$ be a form of top degree. One can write $\omega = f dx^1 \wedge \ldots \wedge dx^n$ for some function $f \in C^\infty(U)$.

Definition. The integral of ω over U is

$$\int_U \omega = \int_U f dx^1 \dots dx^n,$$

where the right hand side is the usual integral. [6]

Problem 11 Assume $F\colon U' \to U$ is a smooth invertible map. Check that

$$\int_U \omega = \int_{U'} F^*\omega$$

is equivalent to the change of variables formula.

[4] i.e. $\omega = \sum_I \alpha_I dx^{i_1} \wedge \ldots \wedge dx^{i_m} + \sum_J \beta_J dt \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_{m-1}}$, where α_I, β_J are smooth functions in $t, x^1, \dots, x^n$ and I, J run over multiindexes $\{i_1 < \ldots < i_m\}$ and $\{j_1 < \ldots < j_{m-1}\}$.

[5] One says that H is a *homotopy* between G_1^* and G_0^*.

[6] A bit of physical terminology from classical mechanics. Given a configuration space M, the (generalized) *velocity* at $p \in M$ is a tangent vector in $T_{p \in M}$. A *force applied to* $p \in M$ is a 1-form in $T^*_{p \in M}$: the force measures the work along different directions (because the concept of energy is more fundamental than that of force). The *work* of the force ν along the given path $\gamma\colon [0,1] \to M$ is the integral $\int_{[0,1]} \gamma^*\nu$. The force ν is *conservative* if $d\nu = 0$: its integrals along closed loops are trivial. A *potential* of ν is a function U such that $dU = \nu$.