Calculus II, CMI. *

Linear algebra preliminaries.

Below $k = \mathbb{R}$ is our base field.

Problem 1 Assume $v_i \in V, i \leq n$ and $\nu^j \in V^*, j \leq m$ is a collection of vectors and covectors, respectively. Let $k\langle v_i\rangle \subset V$ be the span of v_i, i.e. the vector subspace generated by v_i. Consider $\nu^j(v_i)$ as a matrix of the size $n \times m$ [1]. Show that

1. The inequality $\dim k\langle v_i\rangle \geq \operatorname{rk} \nu^j(v_i)$ holds.
2. If $\operatorname{rk} \nu^j(v_i) \geq n$, then v_i are linearly independent.
3. If $\operatorname{rk} \nu^j(v_i) \geq \dim V$, then v_i span V.

Hint: For any matrix A, interpret $\operatorname{rk} A$ as $\dim \operatorname{im}(A)$ and consider the map $V \to k^m$ with j's component equal to $v \to \nu^j(v)$.

Problem 2* *(Dehn's invariant)* Assume $A, B \in \mathbb{R}_{>0}$. Suppose that the rectangle of size $A \times B$ is cutted into a finite number of squares. We say that that $A \times B$ admits a *tiling* by squares. Prove that the ratio A/B is rational.

Problem 3* Suppose the rectangle $A \times B$ admits a tiling by a finite number of rectangles $A_i \times B_i$ such that either $A_i \in \mathbb{Z}$, or $B_i \in \mathbb{Z}$. Prove that either of the sides A, B is rational. [2]

Hint: Treat $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Given a finite rectangular tiling into pieces $A_i \times B_i$ show that $\sum_i A_i \otimes_{\mathbb{Q}} B_i = A \otimes_{\mathbb{Q}} B$.

Problem 4 Assume S is a set. Denote by $k\langle S\rangle$ the vector space with basis S. Check that

1. $k\langle S'\rangle \oplus k\langle S''\rangle = k\langle S' \sqcup S''\rangle$, where $S' \sqcup S''$ is the disjoint union of sets.
2. $k\langle S'\rangle \otimes k\langle S''\rangle = k\langle S' \times S''\rangle$, where $S' \times S''$ is the direct product of sets.

Below $\operatorname{Hom}_k(V, W)$ denotes the vector space of all linear maps $\{f\colon V \to W \mid f \text{ is linear}\}$.

Problem 5 Recall that the identification $V^* \otimes V \simeq \operatorname{Hom}_k(V, V)$ maps $\nu \otimes v \in V^* \otimes V$ to the linear map $w \to \nu(v)w$. Check that

1. Under the identification, the trace map $\operatorname{tr}\colon \operatorname{Hom}_k(V, V) \to k$ corresponds to the pairing map $V^* \otimes V \to k$ given by $\nu \otimes v \to \nu(v)$ for $\nu \in V^*, v \in V$.
2. Assume $A \in \operatorname{Hom}_k(V, V)$ is given by the matrix $A^i{}_j$ with respect to a basis $e_i \in V$ [3]. Let $e^j \in V^*$ be the dual basis of e_i. Then the tensor corresponding to A is $\sum_{ij} A^i{}_j e^j \otimes e_i$.

Problem 6 Assume $A \in \operatorname{Hom}_k(V, V)$ and $B \in \operatorname{Hom}_k(W, W)$. Write down $\operatorname{tr}(A \otimes B)$ and $\det(A \otimes B)$ in terms of $\operatorname{tr}(A), \operatorname{tr}(B)$ and $\det(A), \det(B)$, respectively.

Hint: Use the fact that over $\mathbb{C}$ the set of all diagonalizible matrices is dense in all matrices. Show that it is enough to solve the problem for diagonal matrices over $\mathbb{C}$.

Problem 7 *(Darboux's theorem at one point)* Assume the multilinear map $\omega\colon V \times V \to k$ satisfy two conditions: (1) ω is skew-symmetric: $\omega(v, v) = 0$, (2) ω is non degenerate: $\omega(v, -) = 0$ iff $v = 0$ for all $v \in V$. The corresponding tensor $\omega \in V^* \otimes V^*$ is called *symplectic form* on V.

Show that V has an even dimension $2m$ and there is a basis $e_i, f_j \in V, i, j \leq m$ such that $\omega(e_i, f_j) = \delta_{ij}$ [4] and $\omega(e_i, e_j) = 0$, $\omega(f_i, f_j) = 0$ for all i, j.

In other words there is an identification $F\colon k^{2m} \to V$ such that $F^*\omega := \omega(F-, F-)$ has the form indicated above; in particular all symplectic forms on V are equivalent.

Problem 8 Given $v \in V$, define the *contraction* map $\iota_v\colon \Lambda^m V^* \to \Lambda^{m-1} V^*$ by setting

$$(\iota_v \omega)(v_1, \ldots, v_{m-1}) = \omega(v, v_1, \ldots, v_{m-1})$$

for each $\omega \in \Lambda^m V^*$. We call $|\omega| = m$ the *degree* of ω. Check that

1. The contraction ι_v satisfies the (super-)Leibniz rule: $\iota_v(\omega_1 \wedge \omega_2) = \iota_v(\omega_1) \wedge \omega_2 + (-1)^{|\omega_1|}\omega_1 \wedge \iota_v(\omega_2)$. Thus ι_v is a (super-) *derivative* of the algebra ΛV^*.
2. Two contractions (super-)commute: $[\iota_v, \iota_w] := \iota_v \circ \iota_w + \iota_w \circ \iota_v$ vanish. [5]

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[1] i.e. with n columns and m rows

[2] In fact either A or B is integer, the proof is similar.

[3] Here i runs over the rows and j over the columns of the matrix: the image of the j'th basis vector is given by j'th column of A.

[4] The Kronecker's delta symbol δ_{ij} denotes the identity matrix

[5] One can think of the algebra of skew-symmetric (i.e. alternating) tensors ΛV^* as the *odd* version of the algebra SV^* of symmetric tensors, where $S^m V^*$ is the space of polynomial functions on V of degree m. There is no standard terminology here: the words *graded commutative-/super-* as well as *anti-/skew-* or *even/bosonic*, *odd/fermionic* are in common use.