Tutorial 6- Monday 13/10/2022

We will assume that all rings are nonzero, are commutative and have the unit element.

1. Let $R = \mathbb{Z}[\sqrt{-5}]$. Let $\alpha = \sqrt{-5}$. Let $\mathfrak{p}_1 = (2, 1+\alpha)$, $\mathfrak{p}_2 = (2, 1-\alpha)$, $\mathfrak{p}_3 = (3, 1+\alpha)$ and $\mathfrak{p}_4 = (3, 1-\alpha)$.
   (a) Show that $\mathfrak{p}_i$ are prime ideal for $i = 1, 2, 3, 4$. Are they maximal ideals?
   Show that 2 ,3, $1+\sqrt{-5}$, $1-\sqrt{-5}$ are irreducible in R.
   (b) Show that $(6) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4$.
   (c) Is $(6) = \mathfrak{p}_1^2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4$.
   (d) Determine which of the following ideals are prime ideals:
   (a) $(3+5\alpha)$, $(2+2\alpha)$, (b) $(4+\alpha, 1+2\alpha)$
   (e) Factor (14) into prime ideals in R.
   (f) Determine whether 11 is irreducible in R.
   (g) Let $I = (3, 1+\alpha)$. Show that I^2 is a principal ideal.
2. Let $\delta = (1+\sqrt{-19})/2$ and $R = \mathbb{Z}[\delta]$.
   (a) Show that $R \cong \mathbb{Z}[T]/(T^2 - T + 5)$.
   (b) Determine whether 2 and 3 are prime ideals in R.
   (c) Determine whether the following ideals are (i) prime ideals (ii) maximal ideal in $R[x]$:
   (a) $(7, \delta^3 x + x + 1)$ (b) $(2\delta^2 + 3)$ (c) $(\delta x + 1)$ (d) $(\delta x + 2)$ (e) $(2x\delta + 1)$ (f) $(11, 2x\delta + 1)$
3. Let $R = \mathbb{Z}[\sqrt[3]{2}]$. Is it possible to write (5) as a product of prime ideals?
4. Let $R = \mathbb{Z}[\sqrt{n}]$ where $n \geq 3$.
   (a) Which of the following are (i) irreducible (ii) prime in R?
   (a) 2, (b) $\sqrt{-n}$, (c) $1+\sqrt{-n}$
   (b) Show that R is not a UFD.
5. Let $R = \mathbb{Z}[\omega]$ where $\omega = e^{2\pi i/3}$. Let $p \neq 3$ be a prime.
   (a) Show that $x^2 + x + 1$ has a root in $\mathbb{Z}/p\mathbb{Z}$ if and only if $p \ncong 1 \bmod 3$.
   (b) Show that (p) is maximal ideal if and only if $p \cong -1 \bmod 3$.
   (c) p factors in R if and only if $p = a^2 + ab + b^2$ for some integers a and b.
6. Let $\delta = \sqrt{-3}$ and $R = \mathbb{Z}[\delta]$.
   (a) Let $I = (2, 1+\delta)$, $\overline{I} = (2, 1-\delta)$. Is $I\overline{I}$ a principal ideal.
   (b) Determine whether the ideal $(3x+1)$ is a prime ideal in R.
7. Let $\delta = \sqrt{-6}$ and $R = \mathbb{Z}[\delta]$.
   (a) Determine whether $(2, \delta)$ and $(3, \delta)$ are prime ideals.
   (b) Factor $(6)R$ into prime ideals.