Tutorial 3- Monday 25/8/2022

We will assume that all rings are commutative and have the unit element.

1. Let R be a Euclidean ring. Show that for any $a, b \in R$, there exists an element r such that $(r) = (a, b)$.
2. Show that $\mathbb{Z}[i]$ is a Euclidean domain.
3. Let $R = \mathbb{Z}[\sqrt{10}]$. Let $N : R \to \mathbb{Z}$ be given by $N(a + b\sqrt{10}) = a^2 - 10b^2$.
   (a) Show that $N(uv) = N(u)N(v)$ for all $u, v \in R$.
   (b) Show that $N(u) = 0$ if and only if $u = 0$.
   (c) $N(u) = 1$ if and only if u is a unit.
   (d) $4 + \sqrt{10}$, is irreducible in R but not a prime element.
4. Let $R = \mathbb{R}[x, y]$ and $S = \mathbb{R}[x, y, x/y]$. Let $I = (x^3, xy^2, y^3)$. Let $\phi : R \to S$ be the inclusion map. Prove or disprove: $I^{ec} = I$ (5 marks)
5. Find the gcd of the following elements: (i) $3 + 4i$ and $4 - 3i$ (ii) $11 + 7i$ and $18 - i$.
6. Let R be an integral domain. Suppose R is not a field. Let $U(R)$ denote the set of units in R. Show that $R \setminus \{U(R) \cup \{0\}\} \neq \emptyset$. Suppose that there exists no $y \in R \setminus \{U(R) \cup \{0\}\}$, such that for all $x \in R$, $y|(x - r)$ for some $r \in \{U(R) \cup \{0\}\}$. Show that R is not a Euclidean domain. (5 marks)
7. Show that the following rings are Euclidean domains: (i) $\mathbb{Z}[i]$ (ii) $\mathbb{Z}[\sqrt{-2}]$
8. Determine all the untis in $\mathbb{Z}[\sqrt{2}]$.
9. Show that $\mathbb{Z}[(1 + \sqrt{-19})/2]$ is a PID but not a ED.