Tutorial 3- Monday 25/8/2022

We will assume that all rings are commutative and have the unit element.

1. Let $p \in \mathbb{Z}$ be a non-zero prime. Consider $R := \{a + bi + cj + dk | i^2 = j^2 = -1, ij = -ji = k\}$ be the ring of quartenions over $\mathbb{Z}/p\mathbb{Z}$.
   (a) Show that R has only finitely many elements. Determine this number.
   (b) Show that R is a ring, but not a division ring.
2. Let $R = \mathbb{C}[x, y, z]$ and $S = \mathbb{C}[t]$ where x, t, z, t are variables. Let $\phi : R \to S$ be a homomorphism given by $\phi(x) = t^3$, $\phi(y) = t^4$ and $\phi(z) = t^5$. Show that $ker(\phi) = (x^3 - yz, y^2 - xz, z^2 - x^2y)$.
3. Let $\phi : R \to R'$ be a homomorphism of R into R'. Show that $ker(\phi) := \{a \in R | \phi(a) = 0\}$ is an ideal in R.
4. 'Let R be the set of all real valued functions on $[0, 1]$ and S a subset of $[0, 1]$. Show that the set of all functions on $f \in R$ such that $f(x) = 0$ for all $x \in S$ is a maximal ideal in R.
5. Let p be a prime number, $R = \mathbb{Z}[\sqrt{p}] := \{a + b\sqrt{p} : a, b \in \mathbb{Z}]$ and $S = \mathbb{Z}/p\mathbb{Z}$. Let $\phi : R \to S$ be given by $\phi(a + b\sqrt{p}) = a + p\mathbb{Z}$.
   (a) Show that ϕ is a homomorphism.
   (b) Determine $ker(\phi)$
6. Let R be a ring. An element e is said to be idemopotent if $e^2 = e$
7. Show that $R \cong (e)R \times (1 - e)R$
   (a) Suppose $e_1, \ldots, e_n$ are idempotent elements. Assume in addition they are orthogonal, i.e., and $e_j e_j = 0$ for all $i \neq j$. Let $R_1, \ldots, R_n$ be rings with identity. Show that the following are equivalent:
      i. $R \cong R_1 \times R_2 \times \cdots R_n$
      ii. There exists orthognal idempotent elements $e_1, \ldots, e_n$ such that $e_1 + \cdots + e_n = 1_R$ and $(e_i)R \cong R_i$.
      iii. $R \cong I_1 \times I_2 \times \cdots I_n$ where $I_1, \ldots, I_n$ are ideals in R.
8. Let R be a ring and $I_1, \ldots, I_n$ ideals in R.
   (a) Suppose $I_1 + \cdots I_n = R$ and for each $1 \leq k \leq n$, $I_k \cap (I_1 + \cdots + I_{k-1} + I_{k+1} + \cdots + I_n) = 0$. The show that there is an isomorphism $R \cong I_1 \times \cdots \times I_n$.
   (b) Suppose $I_i + I_j = R$ for all $i \neq j$. If $b_1, \cdots, b_n \in R$, then show that there exists $b \in R$ such that $b \cong b_i \ mod(I_i)$ for all $i = 1, \ldots, n$.
   (c) If $I_1, \ldots, I_n$ are ideals in R, then show that there is an injective map $\phi : R/(I_1 \cap \cdots \cap I_n) \to (R/I_1) \times \cdots \times (R/I_n)$.
9. Let R be a commutative ring with 1 and S a subset or R with the property that (i) $0 \notin S$ (ii) $s_1 s_2 \in S$ implies that $s_1 s_2 \in S$. Let $a_1, a_2 \in R$ and $s_1, s_2 \in S$. Define a relation $\sim$ in $R \times S$ by $(a_1, s_1) \sim (a_2, s_2)$ if there exists $s_3 \in S$ such that $s_3(a_1 s_2 - a_2 s_1) = 0$.
   (a) Show that $\sim$ is an equivalence relation.
   (b) Let R_S denote the set of all equivalence classes and let $[a, b]$ be an element of R_S. In R_S define $[a, b] + [c, d] := [ad + bc, bd]$ and $[a, b] \cdot [c, d] = [ac, bd]$. Show that these are well defined operations.
      i. Show that the mapping $\phi : R \to R_S$ defined by $\phi(a) = [as, s]$ is a homomorphism. Determine the kernel of ϕ.
      ii. Show that $ker(\phi)$ has no element of S in it.
      iii. Let $s_1, s_2 \in S$. Show that every element of the form $[s_1, s_2]$ is invertible.
10. Let $R := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.
    (a) Show that R (i) is a ring (ii) R is a field
    (b) What happens if $\mathbb{Q}$ is replaced by $\mathbb{Z}$.
11. Let R, S be rings and $\phi : R \to S$ be a homomorphism of rings.
    (a) Suppose ϕ is surjective.
       i. Show that for every ideal I in R, $\phi(I)$ is an ideal in S.
       ii. Show that if P is a prime ideal in R, then $\phi(P)$ is a prime ideal in S.
    (b) If Q is a prime ideal in S, then show that $\phi^{-1}(Q)$ is a prime ideal in R which contains $ker(\phi)$.
    (c) Descirbe all prime ideals in $R = \mathbb{Z}$. Describe all prime ideals in $S = Z[x]$.