Problem set 2 , August 18, 2025

Tutorial 3- Monday 8/8/2022

We will assume that all rings have the unit element.

1. Let $U(R)$ denote the set of units in R. Show that $U(R)$ is a group with respect to multiplication. If R is an integral doman and x is a variable, then show that $U(R[x]) = U(R)$.

2. Let $n \geq 1$ be an integer. Determine $U((\mathbb{Z}/n\mathbb{Z})[x])$ and $U(\mathbb{Z}/n\mathbb{Z})$.

3. Determine the kernel and image of the map in the following cases:

   (a) Let x, y, t be variables, a, b be positive integers with $gcd(a, b) = 1$. Let $\phi : \mathbb{C}[x, y] \to \mathbb{C}[t]$ be the homomorphism given by $\phi(x) = t^a$ and $\phi(y) = t^b$.

   (b) Let x be a variable and $\phi : \mathbb{Z}[x] \to Z[i]$ be given by $\phi(r + sx) = r + si$.

   (c) Let R be a ring of characteristic p. Consider the map $\phi : R \to R$ given by $\phi(r) = r^p$. Show that ϕ is a ring homomorphism.

4. Let R be an integral domain. Let $S(R) := \{(a, b) : a, b \in R, b \neq 0\}$. Consider the relation $\sim$ on $S(R)$ give by $(a, b) \sim (c, d)$ if and only if $ad = bc$.

   (a) Show that $\sim$ is an equivalence relation.

   (b) Define $Q(R) := S(R)/\sim$. Let — denote image in $Q(R)$. Define operations on $Q(R)$ as follows:

   $$\begin{aligned} \overline{(a,b)} + \overline{(c,d)} &= \overline{(ad+bc, bd)} \\ \overline{(a,b)} \circ \overline{(c,d)} &= \overline{(ac, bd)}. \end{aligned}$$

   Show that sum and product are well defined. Show that with these operations show that $Q(R)$ is a field. What is the inverse of $\overline{(a, b)}$.

5. Let R be a subset of rationals m/n such that $n \neq 0$ and p does not divide n. Show that R is a ring.

6. Let R be a ring. Let $Z := \{a \in \mathrm{R}\}$ such that $ax = xa$ for all $x \in R$. Show that Z is a subring of R. (Z is called the center of R).

7. Show that a ring has no zero divisors if and only if right and left cancellation laws hold. i.e., if $ac = bc \implies b = c$ (similarly if $ca = ba$).

8. Let $R = \mathbb{Z}$. Determine all prime ideals in $\mathbb{Z}[x]$.