Tutorial 9- Monday 17/11/2025

1. Let $F = \mathbb{F}_8$. Determine all the elements of $F$. Find the irreducible polynomial of each of the elements.
2. Find the 13-th root of 2 in the field $\mathbb{F}_8$.
3. Factor $x^9 - x$ and $x^{27} - x$ in $\mathbb{F}_3$.
4. Factor the polynomial $x^{16} - x$ over the field's $\mathbb{F}_4$ and $\mathbb{F}_8$.
5. Let $K$ be a finite fie ld. Prove that the product of the nonzero elements of $K$ is $-1$.
6. The polynomials $f(x) = x^3 + x + 1$ and $g(x) = x^3 + x^2 + 1$ are irreducible over $\mathbb{F}_2$. Let $K$ be the field extension obtained by adjoining a root of $f$ , and let $L$ be the extension obtained by adjoining a root of $g$. Describe explicitly an isomorphism from $K$ to $L$, and determine the number of such ismorphisms.
7. Let $F = \mathbb{F}_p$.
   (a) Determine the number of monic irreducible polynomials of degree 2 in $F[x]$.
   (b) Let $f(x)$ be an irreducible polynomial of degree 2 in $F[x]$. Prove that $K = F[x]/f(x)$ is a field of order $p^2$, and that its elements have the form $a + b\alpha$, where $a$ and $b$ are in $F$ and $\alpha$ is a root of $f(x)$ in $K$. Moreover, every such element with b=t-Q is the root of an irreducible quadratic polynomial in $F[x]$.
   (c) Show that every polynomial of degree 2 in $F[x]$ has a root in $K$.
   (d) Show that all the fields $K$ constructed as above for a given prime $p$ are isomorphic .
8. Let $F$ be a finite fie ld, and let $f(x)$ be a non constant polynomial whose derivative is the zero polynomial. Prove that $f(x)$ can not be irreduc ible over $F$ .
9. L e t $f(x) = ax^2 + bx + c$ with $a, b, c$ in a ring $R$ . Show that the ideal of the polynomial ring $R[x]$ that is generated by $f(x)$ a n d $f'(x)$ contains the discriminant, the constant polynomial $b^2 - 4ac$.
10. Let $p$ be a prime integer , and le t $q = p^r$ and $q' = p^k$. For which values of $r$ and $k$ does $x^{q'}x$ divide $x^q - x$ in $\mathbb{Z}[x]$?
11. Prove that a finite subgroup of the multiplicative group of any field $F$ is a cyc lic group .