Tutorial 6- Monday 17/10/2022

Throughout $K \supset F$ is a field extension.

1. Let $F_1 \subset F_2 \subset F_3$ be fields. Show that $[F_3 : F_1] = [F_3 : F_2][F_2 : F_1]$.
2. Show that $\pi \cdot i$ is algebraic over $\mathbb{R}$. Is it algebraic over Q?
3. Show that $\sqrt{2} + \sqrt{3}$ is algebraic over $\mathbb{Q}$. Compute $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}]$.
4. Determine the splitting field over $\mathbb{Q}$ for the following: (a) $x^4 - 2$ (b) $x^4 + 2$.
5. Show that $[K : F]$ is prime then there are no intermediate fields between K and F.
6. Let $\alpha_1, \ldots, \alpha_n \in K$. Show that $K(\alpha_1, \ldots, \alpha_n) = K(\alpha_1, \ldots, \alpha_n)$.
7. For any subset $S \subseteq K$, let $F(S)$ denote the smallest field containing F and S. If $K_1, \ldots, K_n$ be subfields of K, then show that $K_1(K_2(\ldots K_{n-1}(K_n))) = K_1 \cdots K_n$.
8. Let $\alpha \in K$ be algebraic of odd degree. Show that $K(\alpha) = K(\alpha^2)$.
9. Let $x^n - a \in F[x]$ be irreducible and $\alpha \in K$ a root of $x^n - a$. Suppose m divides n. Determine the degree of the irreducible polynomial satisfied by α^m.
10. K is algebraic over F, L is an integral domain such that $F \subseteq L \subseteq K$. Show that L is a field.
11. Let $F = \mathbb{Q}$ and $f(x) = x^5 + 2x + 2 \in F[x]$.
    (a) Show that $f(x)$ is irreducible.
    (b) Let α be a real root of $f(x)$ and $K = \mathbb{Q}(\alpha)$ and let $\{1, \alpha, \ldots, \alpha^4\}$ be a basis of K over $\mathbb{Q}$. Express $(\alpha^2 + 2)(\alpha^3 + 3\alpha)$ in terms of the basis elements.
12. Let F be a finite field and let $f(x) \in F[x]$ be a non-constant poluynomial whose derivate is the zero polynomial. Show that $f(x)$ cannot be irreducible over F.