Tutorial 7- Monday 27/10/2022

1. Let $R \subseteq S$ be an integral domains and let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Define $f'(x) = \sum_{i=i}^{n} a_i k x^{i-1}$. Let $a \in S$.
   (a) Show that a is a multiple root of f if and only if $f(a) = 0$ and $f'(a) = 0$.
   (b) Let R be a field and suppose $gcd(f, f') = 1$. Show that f has no multiple roots in S.
   (c) Let R be a field and suppose $f(x)$ is irreducible in $R[x]$ and suppose there is a root of f in S. Show that f has no multiple roots in S if and only if $f' \neq 0$.
2. Show that $1 + x + \cdots + x^{p-1} \in \mathbb{Q}[x]$ is irreducible.
3. Let p be a prime, $q = p^e$ and $r = p^{e-1}$. Show that $(x^q - 1)/(x^r - 1)$ is irreducible.
4. Let $f, g \in \mathbb{Z}[x]$. Show that f and g are relatively prime in $\mathbb{Q}[x]$ if and only if the ideal they generate in $\mathbb{Z}[x]$ contains an integer.
5. Let $R = \mathbb{Z}[x_1, x_2, x_3x_4]$. Determine which of the following polynomials are irreducible: (i) $f = x_1^{p+1} - x_2^p$ (ii) $g = x_1x_4 - x_2x_3$(iii) $h = x_1^p x_2 - x_3^p$ (iv) $fh + g^p$.
6. Determine which of the following polynomials in $\mathbb{C}[x, y, z]$ are irreducible: (1) $y^5 - z^4$ (2) $z^8 + 2xy^3 - x^2yz - z^3$ (3) $2y^2z^6 + xz^7 - x^3y - 2y^2z + 3xz^2$ (4) $y^4z^5 - y^4 + 2xy^2z - x^2z^2$ (5) $2xy^4z^4 + yz^7 + 3x^2y^2 - 2x^3z - yz^2$ (6) $x^2y^4z^3 + 3y^3z^5 + 2xyz^6 - x^4 - 3y^3 + 4xyz$
7. Show that $x^p - x - c$ is irreducible in $\mathbb{Z}/p\mathbb{Z}[x]$ if and only if it has no root in $\mathbb{Z}/p\mathbb{Z}$.