Homework 3

1. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$. Is A diagonalizable? If yes, diagonalize it explicitly, i.e., find P such that ...

2. (From SMMC 2023) Let n be a positive integer. Let A, B and C be three n-dimensional subspaces of $\mathbb{R}^{2n}$ with the property that $A \cap B = B \cap C = C \cap A = \{0\}$. Prove that there exists a basis $\{a_1, a_2, \ldots, a_n\}$ of A, a basis $\{b_1, b_2, \ldots, b_n\}$ of B and a basis $\{c_1, c_2, \ldots, c_n\}$ of C with the property that for each $i \in \{1, 2, \ldots, n\}$, the vectors a_i, b_i and c_i are linearly dependent.

3. The set of invertible $n \times n$ matrices with entries in a field F is denoted $GL_n(F)$ and called the general linear group. Observe that the number of ordered bases of F^n is in bijection with $GL_n(F)$.

(a) Show that $|GL_2(\mathbb{F}_p)| = p(p+1)(p-1)^2$. Find $|GL_n(\mathbb{F}_p)|$. (Optional: Find $|SL_n(\mathbb{F}_p)|$, i.e., count matrices with determinant 1. The key point is that the determinant map from $GL_n(F)$ has uniform fibers. Why?)

(b) How many subspaces of each possible dimension are there in $\mathbb{F}_p^3$?

4. (a) Let P be a linear operator on V. Show that $P^2 = P$ if and only if $P|_{image(T)} =$ identity on $image(T)$. Such a P is called a projection (NOT orthogonal projection: that requires more structure). Is a projection diagonalizable? What is its trace?

(b) Projections are useful because they closely related to breaking up a vector space as a direct sum. Show that $V = K \oplus W$ if and only if there is a projection P such that $ker(P) = K$ and $image(P) = W$. (Compare Artin's Proposition 4.3.1b.)

Optional but important addendum: Projections are also closely related to direct sum decomposition into *invariant* subspaces. Suppose T is a linear operator on V and K is a T-invariant subspace of V.

Question: Can we write $V = K \oplus W$ with W also T-invariant? In other words, can we arrange T to have a 2×2 block *diagonal* matrix rather than just a block *triangular* matrix? That would break the action of T into two independent parts, which would clearly help in trying to understand T.

Answer: this is possible if and only if there is a projection P with $ker(P) = K$ and $TP = PT$. Prove this.

Further remarks: Operators commuting with T are interesting. E.g., see that the kernel and image of such an operator are T-invariant subspaces. Everything in the addendum becomes transparent when expressed in the language of modules over an appropriate ring.

5. (a) For a vector space V with basis $\{u, v, w\}$, consider the linear operator T defined by $T(u) = v, T(v) = w$ and $T(w) = u$. Write the matrix A of T with respect to the given basis. Take the underlying field to be in turn $\mathbb{R}, \mathbb{C}$ and $\mathbb{F}_p$ where p is a prime number. For each of these cases, decide if T is diagonalizable. For $\mathbb{F}_p$ your answer may be in terms of a condition on the prime p.

(b) A 3×3 matrix B with entries in $\mathbb{C}$ satisfies $tr(B) = tr(B^2) = 0$ and $tr(B^3) = 3d$ for some given *nonzero* scalar d in $\mathbb{C}$. Find $det(B)$. Show that all such matrices are similar to each other.

6. Recall that to analyze a linear operator T on a vector space V, it may help to decompose V as a direct sum of T-invariant subspaces. (a) Solve problem 5.9 in chapter 4. (b) Then solve problem 5.10.

7. (a) For a diagonalizable operator T on V, consider the eigenspace decomposition $V = \oplus_\lambda V_\lambda$. Show that any T-invariant subspace U of V is of the form $\oplus_\lambda W^\lambda$, where W^λ is a subspace of V_λ. There is something to prove here! Is $T|_U$ diagonalizable?

(b) Suppose T and S are both diagonalizable operators on a finite dimensional vector space V. Show that if $TS = ST$ then there is a single basis of V with respect to which both S and T have diagonal matrices. The key is to consider an entire eigenspace of T from the point of view of S. (Quick: is the converse true?) Now show that any family (even infinite) of mutually commuting operators on V can be simultaneously diagonalized.

8. The Cayley-Hamilton theorem states that over any field, any linear operator T on a finite dimensional vector space satisfies its own characteristic polynomial $p_T(x)$, i.e., $p_T(T) =$ the 0 operator. We sketched this in class first for diagonal(izable) operators and then for triangular(izable) operators. It may help to review that proof. Now prove the result for arbitrary T using the outline below. I learned this proof from Vighnesh.

(a) Solve problem 5.6 in chapter 4. Show that $p_T(T)(v_0) = 0$ vector. Deduce quickly that $p_T(T) = 0$ operator.

(b) Let T be any operator on V. Start with any nonzero vector v_0 and generate a T-invariant subspace U_1 as in part (a) with basis $v_0, \ldots, v_n$. If $U_1 = V$ we are done. Otherwise take a vector w_0 outside U_1. Keep applying T as long as you get vectors linearly independent of ALL previously chosen basis vectors including those in U_1. Name the resulting T-invariant span of $v_0, \ldots, v_n, w_0, \ldots, w_m$ as U_2. What is the characteristic polynomial of $T|_{U_2}$? Show that $T|_{U_2}$ satisfies its own characteristic polynomial. Continue until one exhausts V. You need not make careful notation for U_3 onwards but clearly explain the general pattern in words.

(c) Is the result in problem 5(b) true if we replace every $\mathbb{C}$ by $\mathbb{R}$? (So similarity is also required to be over $\mathbb{R}$.)