Analysis 1: Mid-semester Examination

Date: September 23, 2026 **Time:** 3 hours **Marks:** 100

Answer *any four* questions. Each question carries 25 marks.

Write the answer to each question on a separate set of sheets. Write your name and roll number on each set.

You may use definitions of limit of a sequence, limsup and liminf of a sequence; that there exists a subsequence converging to the limsup; ratio and root tests for convergence of a series; definitions of continuity and uniform continuity; results related to sequences; continuity of a function. In problems related to continuity/uniform continuity you can use results related to sequences after stating them precisely, not by their name. You can also use any part of any question asked in the mid-sem while answering any other part, even if you have not answered the part you are quoting. Anything else, you have to write a proof.

I. Let $(a_n : n \geq 1) \subseteq \mathbb{R}$ be a sequence and let $s_n = \sum_{k=1}^{n} a_k$.

[5] (a) If $(s_n : n \geq 1)$ converges to $s \in \mathbb{R}$, show that $(a_n : n \geq 1)$ converges to zero.

[8] (b) If $a_n = (-1)^n b_n$, where b_n satisfies $0 \leq b_{n+1} \leq b_n$ for all $n \geq 1$, show that s_n converges.

[12] (c) Let $0 \leq a_{n+1} \leq a_n$ for all n. For $m \geq 1$ let $b_m = 2^m a_{2^m}$ (thus $b_3 = 8a_8$). Show that the series $\sum_m b_m$ is convergent if and only if the series $\sum_n a_n$ is convergent.

[25] II. Let $(a_n : n \geq 1) \subseteq \mathbb{R}$ be a sequence such that

(a) for any subsequence $(a_{n_k} : k \geq 1)$ of $(a_n : n \geq 1)$, writing $b_k = a_{n_k}$, there is a subsequence $(b_{k_j} : j \geq 1)$ of $(b_k : k \geq 1)$ such that $c_j = b_{k_j}$ converges;

(b) there is $\alpha \in \mathbb{R}$ such that for all subsequences $(a_{n_k} : k \geq 1)$ of $(a_n : n \geq 1)$ that converge, $(a_{n_k} : k \geq 1)$ converges to α.

Show that $(a_n : n \geq 1)$ converges to α.

III. Let $f : (0, 1) \to \mathbb{R}$ be a function.

[10] (a) If f is continuous, show that for any sequence $(a_n : n \geq 1) \subseteq (0, 1)$ converging to $a \in (0, 1)$, the sequence $(f(a_n) : n \geq 1)$ converges to $f(a)$.

[15] (b) Suppose that for any sequence $(a_n : n \geq 1) \subseteq (0, 1)$ converging to $a \in (0, 1)$, the sequence $(f(a_n) : n \geq 1)$ converges to $f(a)$. Show that f is continuous.

IV. Let $\{a_n : n \geq 0\}$, $\{b_n : n \geq 0\}$ be sequences such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are convergent for all $x \in \mathbb{R}$.

[6] (a) Show that f, g are continuous functions on $\mathbb{R}$.

[7] (b) Let $h(x) = f(x)g(x)$ for $x \in (-1, 1)$. Show that h also admits a power series expansion $h(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in \mathbb{R}$. Find an expression for c_n, $n \geq 0$, in terms of $\{a_n\}$ and $\{b_n\}$.

[8] (c) Let $e(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$. Show that e is a continuous function and that $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

[4] (d) Show that $e(x) > 0$ for all $x \in \mathbb{R}$.

V. Let $f, g: \mathbb{R} \to \mathbb{R}$ be continuous functions.

[7] (a) Let $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Show that h is a continuous function.

[10] (b) Show that for all $a, b \in \mathbb{R}$ with $a < b$, and $0 < \alpha < 1$, there is $c \in (a, b)$ such that $f(c) = \alpha f(a) + (1-\alpha) f(b)$.

[8] (c) Suppose $\lim_{x\to\infty} f(x) = u \in \mathbb{R}$ and $\lim_{x\to-\infty} f(x) = v \in \mathbb{R}$. Show that f is bounded and also uniformly continuous on $\mathbb{R}$.