Analysis 1: Aug-Nov 2026
Group Project

Dear Students,
I am giving a set of questions as a Group Project:

I have created 3 groups (see below): Each group is to work as a team and solve the problems given below. Each group is to submit a writeup giving the solutions (so only 3 documents are to be submitted).

It is strongly recommended to create this report using LaTeX. Writing in LaTeX is now an essential skill in Mathematics, Physics, and Computer Science, so please take this opportunity to learn from one another. If your group does not have someone who knows how to use LaTeX, do ask around in the hostel. I am sure you will get help.

Submission Deadline: Submit both the report in one week, by **October 14th**.
Presentation Session: A presentation session may be scheduled after **October 20th**.
Individual Understanding: While this is a group effort, **each group member** must thoroughly understand all material and ideas submitted by the group.
Grading: This assignment will **NOT** count directly toward your final grade. However, key concepts and ideas from these projects may be evaluated in future quizzes or the end-semester examination. Since this project has no impact on your grades, so you can use any source: but the group as a whole should take responsibility of final report

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Prove all the assertions from first principles.
You can use results on convergence of sequences; continuity and uniform continuity of functions; root test, ratio test for power series (but you must give precise statement(s)).
Do not use any other standard result(s) without providing proof.

(I) Let $\{a_n : n \geq 0\}$, $\{b_n : n \geq 0\}$ be sequences such that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are convergent for all $x \in (-M, M)$.
[a] Show that f is a differentiable function on $(-M, M)$ and the derivative f' also admits a power series expansion that is convergent for all $x \in (-M, M)$.
[b] Let $h(x) = f(x)g(x)$ for $x \in (-M, M)$. Show that h also admits a power series expansion $h(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (-M, M)$. Find expression for $\{c_n : n \geq 0\}$ in terms of $\{a_n : n \geq 0\}$, $\{b_n : n \geq 0\}$.

(II) For $n \geq 0$, let $u_{2n} = 0$, $v_{2n} = (-1)^n \frac{1}{(2n)!}$, $u_{2n+1} = (-1)^n \frac{1}{(2n+1)!}$, $v_{2n+1} = 0$.

Let $A(x) = \sum_{n=0}^{\infty} u_n x^n$, $B(x) = \sum_{n=0}^{\infty} v_n x^n$.

(a) Show that the power series defining $A(x)$ and $B(x)$ converge for all $x \in \mathbb{R}$.
(b) Show that $A(0) = 0$ and $B(0) = 1$.
(c) Show that the derivative $A'(x)$ of $A(x)$ equales $B(x)$, likewise $B'(x)$ equals $-A(x)$.
(d) Show that $\exists \theta > 0$ such that $A(\theta) = 0$ and $A(x) > 0$ for $x \in (0, \theta)$.
(e) Let $g(x) = (A(x))^2 + (B(x))^2$. Show that $g(0) = 1$ and $g'(x) = 0$ for all x.
(f) Show that $g(x) = 1$ for all x.
(g) Show that $|A(x)| \leq 1$, $|B(x)| \leq 1$ for all x.
(h) Show that $\exists \tau \in (0, \theta)$ such that $A(\tau) = 1$ and $B(\tau) = 0$.
(i) Show that for $x, y \in \mathbb{R}$, $A(x + y) = A(x)B(y) + B(x)A(y)$ and
$B(x + y) = B(x)B(y) - A(x)A(y)$.
(j) Show that for $x \in \mathbb{R}$, $A(x + \tau) = B(x)$ and $B(x + \tau) = -A(x)$.
(k) Show that $\theta = 2\tau$.
(l) Show that A, B are periodic functions with period 2θ.
(m) Show that if $f : \mathbb{R} \mapsto \mathbb{R}$ is any function that is twice differentiable and satisfies the equation $f''(x) = -f(x)$, then $\exists a, b \in \mathbb{R}$ such that $f(x) = aA(x) + bB(x)$ for all x.

As several (if not all) of you could have guessed, $A(x)$ is $\sin(x)$ and $B(x)$ is $\cos(x)$ and $\theta = \pi$.

So try to write various properties of $\sin(x)$ and $\cos(x)$ that you know and see which of these you can prove that A and B satisfy using only the definitions and properties given about and convergence of sequences, power series, continuity, differentiability, ...