Quiz 3

- As usual explain everything fully using complete sentences.
- S, T are operators on finite-dimensional inner product spaces over $F = \mathbb{C}$ or $\mathbb{R}$.
- You may use any definitions and results stated and proved in class (clearly specify what result you are using), but no other definitions or results.

---

1. Let $\mathbb{F} = \mathbb{C}$.

   (a) Given $\langle Sv, w\rangle = 2 + 3i$, calculate $\langle S^*w, v\rangle$.

   (b) Let $f(v) = \langle Tv, v\rangle$. Express $\langle Tu, w\rangle$ in terms of $f(u + w)$, $f(u - w)$, $f(u + iw)$, and $f(u - iw)$. (Patient calculation will work.) Find all operators T for which $f(v) = 0$ for all v.

   (c) Show that if $\langle Tv, v\rangle$ is real for all v, then T is Hermitian.

2. Let

$$M = \begin{pmatrix} 2 & 2 & 3 \\ ? & 5 & 6 \\ ? & ? & 10 \end{pmatrix}.$$

   Suppose $\mathbb{R}^3$ (equipped with the standard inner product) has a basis of mutually perpendicular eigenvectors of M. Construct one such basis. You may start by showing that $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$ is an eigenvector. This is provided only to aid your calculations; your answer should show a method that works without assuming this information.

3. (a) Suppose that the matrix representation of a normal operator T with respect to some orthonormal basis is upper triangular. Prove that this matrix must be diagonal. (You may take a 3×3 matrix to write the argument but indicate how it generalizes.)

   (b) When $F = \mathbb{C}$, part (a) can be used to give another proof of the Spectral Theorem for normal operators. How? You may start with the fact that there is some basis with respect to which the matrix of T is upper triangular. How will you show there is an *orthonormal eigen*basis?