Nice operators on f.dim I.P Sp V over $F = \mathbb{C}$ / $\mathbb{R}$

Def^n ① to ③ are equivalent for operator T on V

	Unitary/orthogonal	Self-adjoint Hermitian/Symmetric	Normal/Real normal
① Eq^n wrt $\langle . \rangle$	$\langle Tv, Tw \rangle = \langle v, w \rangle$	$\langle Tv, w \rangle = \langle v, Tw \rangle$	$\langle Tv, Tw \rangle = \langle T^*v, T^*w \rangle$
② Rel^n with T^*	$T^*T = I \Leftrightarrow T^* = T^{-1} \Leftrightarrow TT^* = I$	$T^* = T$	$TT^* = T^*T$
③ Matrix M wrt any O.N. basis satisfies	$M^*M = I \Leftrightarrow M^* = M^{-1} \Leftrightarrow MM^* = I$ $\Leftrightarrow$ Cols of M are orthonormal $\Leftrightarrow$ rows of M are orthonormal	$M^* = M$	$MM^* = M^*M$

Adjt of an operator T with matrix M wrt an O.N basis $\mathcal{B}$ is the operator, called T^* whose matrix wrt same $\mathcal{B}$ is $M^* := \overline{M}^T$

$\Leftrightarrow \langle Tv, w \rangle = \langle v, T^*w \rangle \Leftrightarrow \langle v, Tw \rangle = \langle T^*v, w \rangle$

Spectral thm over $\mathbb{C}$ "Good operators are orthonormally diag with good evalues"

a) T normal $\iff \exists$ O.N. basis of V consisting of evectors of T with arbitrary evalues

b) T unitary $\iff$ ——————"—————— with evalues of $|\cdot| = 1$

c) T Hermitian $\iff$ ——————"—————— with evalues real.

Pf a) ($\Leftarrow$ easy) of $\Rightarrow$ uses key claim plus induction on $\dim V$.

Key claim ↓

$$\begin{array}{ccc} Tv = \lambda v & \Rightarrow & T^* v = \bar{\lambda} v \\ \Updownarrow & & \Updownarrow \\ (T - \lambda I)v = 0 & & (T^* - \bar{\lambda} I)v = 0 \\ \Updownarrow & & \Updownarrow \\ \|(T - \lambda I)v\| = 0 & \overset{!}{\Rightarrow} & \|(T^* - \bar{\lambda} I)v\| = 0 \end{array}$$

Justification of "!" Let $S = T - \lambda I$ $\quad S^* = T^* - (\lambda I)^* = T^* - \bar{\lambda} I$

Note that S is normal. $SS^* = S^*S$ by inspection

$\therefore \langle Sv, Sv \rangle = \langle S^* v, S^* v \rangle$

Pf of thm: Take T-evector v. Let $W = \mathbb{C}v$. Take O.N. basis: v, ON basis of $W^\perp$

Wrt this matrix of T is of form $\begin{array}{|c|c|}\hline \lambda & 0 \\ \hline 0 & A \\ \hline \end{array}$ b/c $W^\perp$ is T-invariant: Why?

$\therefore$ Matrix of $T^* = \begin{array}{|c|c|}\hline \bar{\lambda} & 0 \\ \hline 0 & A^* \\ \hline \end{array}$

$TT^* = T^*T$ implies $AA^* = A^*A$

$T|_{W^\perp}$ has matrix A, which is normal $\therefore$ By induction $W^\perp$ has an O.N. ebasis for $T|_{W^\perp}$. ☒

b) T unitary $Tv = \lambda v \Rightarrow \langle v, v\rangle = \langle Tv, Tv\rangle = \langle \lambda v, \lambda v\rangle = \bar{\lambda}\lambda\langle v, v\rangle$.

c) T self-adjoint $Tv = \lambda v \Rightarrow \bar{\lambda}\langle v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \lambda\langle v, v\rangle$

Let $u \in W^\perp \iff \langle v, u\rangle = 0$

We want to show $\langle v, Tu\rangle = 0$

Note $\langle v, Tu\rangle = \langle T^*v, u\rangle \overset{!}{=} \langle \bar{\lambda}v, u\rangle$

$= \lambda\langle v, u\rangle = 0$

-OR- directly use claim

(check by writing matrix of T

$\left(\begin{array}{|c|c|}\hline \lambda & \cdots \\ \hline 0 & A \\ \hline \end{array}\right)$ and deduce $v_m^* = 0$

Rmk Matrix forms of these thms

If M is normal/unitary/Hermitian, then $\exists$ a unitary change of basis matrix P ($P^* = P^{-1}$)

such that $P^* M P = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ with λ_i arbitrary / on unit circle / real.

What happens over $\mathbb{R}$? a,b) No spectral thm BUT close! | Exercise

Normal $\Longleftrightarrow$ O.N basis wrt which matrix of normal T is block diag with blocks of size 1×1 or 2×2

AND 2×2 blocks of specific type that you should discover)

Orthogonal $\Longleftarrow$ Same with 2×2 blocks of type $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

c) T symmetric $\Longleftrightarrow \exists$ O.N. ebasis.

Pf Take matrix M of T wrt ON basis. Regarding $v \to Mv$ as Herm op on $\mathbb{C}^n$, we know M has a real evalue. Go back to $\mathbb{R}^n$ and induct, using W T inv $\Rightarrow$ so is $W^\perp$

Def^n A Hermitian/Symmetric matrix M / operator T on a f.dim IP space is called positive (semi) definite if

① $\forall v \neq 0$ $\langle Tv, v\rangle$ is <u>real</u> and > 0 (resp ≥ 0)

i.e $(M[v])^*[v] = [v]^* M^* [v] = [v]^* M [v] > 0$ (resp ≥ 0)

Equivalent cond^n ① $\Leftrightarrow$ ② All evalues of T are > 0 (resp ≥ 0)

$\Leftrightarrow$ ③ $M = P^* P$ with P invertible (resp. any)

$\Downarrow$

$[v]^* M [v] = [v]^* P^* P [v]$

$= (P[v])^* P[v] \geq 0$

For ② $\Rightarrow$ ③: How to find magic P?

Use spectral thm! Take $\sqrt{\cdot}$ diag entries

Rmk

Pos def $\Rightarrow$ invertible

$\Rightarrow$ is easy

$\Leftarrow$ use O.N. ebasis $e_1, \dots, e_n$ and calculate!